Spherical Electromagnetic Waves

The vector nature of the wave fields make the description of the radiation fields quite complicated. Unlike plane waves, the polarization is not a constant, but depends on θ and ϕ. Another difference is that for spherical waves there is a region within a wavelength or so of the origin where the fields contain non-radiative components associated with the character of the source, i.e. dipole, quadrupole, etc.

The easiest description of spherical modes is through the vector operator L and its components, familiar to you from quantum mechanics.

Vector L Operator $\mathbf{L} = -i\cdot(\mathbf{r}\times\nabla)$ L operates only on angular functions

$$\nabla^2 = \frac{1}{r^2}\cdot\frac{\delta}{\delta r}\left(r^2\cdot\frac{\delta}{\delta r}\right) - \frac{L^2}{r^2} \qquad L^2\cdot Y_{lm} = l\cdot(l+1)\cdot Y_{lm}$$

$$L_{up} = L_x + i\cdot L_y = e^{i\cdot\phi}\cdot\left(\frac{\delta}{\delta\theta} + i\cdot\cot(\theta)\cdot\frac{\delta}{\delta\phi}\right) \qquad L_{up}\cdot Y_{lm} = \sqrt{(l+m+1)\cdot(l-m)}\cdot Y_{l,m+1}$$

$$L_{dn} = L_x - i\cdot L_y = e^{-i\cdot\phi}\cdot\left(-\frac{\delta}{\delta\theta} + i\cdot\cot(\theta)\cdot\frac{\delta}{\delta\phi}\right) \qquad L_{dn}\cdot Y_{lm} = \sqrt{(l+m)\cdot(l-m+1)}\cdot Y_{l,m-1}$$

$$L_z = -i\cdot\frac{\delta}{\delta\phi} \qquad L_z\cdot Y_{lm} = m\cdot Y_{lm}$$

Radiation Mode Fields

$e^{-i\cdot\omega\cdot t}$ time dependence, assume the outgoing radial solutions $h_l(k\cdot r) = j_l(k\cdot r) + i\cdot n_l(k\cdot r)$ $k = \frac{\omega}{c}$

Transverse Electric $\mathbf{r}\cdot\mathbf{E} = 0$ $\mathbf{E} = E_0\cdot h_l(k\cdot r)\cdot\mathbf{L}\cdot Y_{lm}(\theta,\phi)$ $\mathbf{B} = -\frac{i}{c\cdot k}\cdot\nabla\times\mathbf{E}$

Transverse Magnetic $\mathbf{r}\cdot\mathbf{B} = 0$ $\mathbf{B} = B_0\cdot h_l(k\cdot r)\cdot\mathbf{L}\cdot Y_{lm}(\theta,\phi)$ $\mathbf{E} = \frac{i\cdot c}{k}\cdot\nabla\times\mathbf{B}$

From the definition of L, you can see that all of Maxwell's equations are satisfied: the divergences because of the $\_\times\nabla$ or $\nabla\times\_$ and the curls by Helmholtz solution and definition.

Examples: Dipole Radiation

There are six linearly independent forms, combinations of p_x, p_y, p_z and m_x, m_y, m_z.. Choose either TM, or TE, with l=1, and m= -1, 0, +1 in the Ylm.

Here the fields are worked out for **dipole TM, m=0**.

Find the vector spherical harmonic...

$$\mathbf{L}\cdot Y_{10} = \frac{1}{\sqrt{2}}\cdot\begin{pmatrix} Y_{1,1}+Y_{1,-1} \\ \dfrac{Y_{1,1}-Y_{1,-1}}{i} \\ 0 \end{pmatrix} = i\cdot\sqrt{\frac{3}{4\cdot\pi}}\cdot\sin(\theta)\cdot\begin{pmatrix} -\sin(\phi) \\ \cos(\phi) \\ 0 \end{pmatrix} = i\cdot\sqrt{\frac{3}{4\cdot\pi}}\cdot\sin(\theta)\cdot\mathbf{e}_{\phi}$$

...use for B...

$$\mathbf{B} = i\cdot\sqrt{\frac{3}{4\cdot\pi}}\cdot B_0\cdot h_1(k\cdot r)\cdot\sin(\theta)\cdot\mathbf{e}_{\phi} = \frac{E_1}{c}\cdot h_1(k\cdot r)\cdot\sin(\theta)\cdot\mathbf{e}_{\phi} \qquad E_1 = i\cdot\sqrt{\frac{3}{4\cdot\pi}}\cdot c\cdot B_0$$

...take the curl for the electric field:

$$\mathbf{E} = \frac{i\cdot c}{k}\cdot\nabla\times\mathbf{B} = \frac{i\cdot E_1}{k}\cdot\left[\mathbf{e_r}\cdot\left[\frac{1}{r\cdot\sin(\theta)}\cdot\frac{\delta}{\delta\theta}\left(\sin(\theta)\cdot h_1(k\cdot r)\cdot\sin(\theta)\right)\right] + \mathbf{e}_{\theta}\cdot\left[-\frac{1}{r}\cdot\frac{\delta}{\delta r}\left(r\cdot h_1(k\cdot r)\cdot\sin(\theta)\right)\right]\right]$$

$$\mathbf{E} = i\cdot E_1\cdot\left[\mathbf{e_r}\cdot 2\cdot\cos(\theta)\cdot\frac{h_1(k\cdot r)}{k\cdot r} - \mathbf{e}_{\theta}\cdot\sin(\theta)\cdot\left(h'_1(k\cdot r) + \frac{h_1(k\cdot r)}{k\cdot r}\right)\right]$$

Now put in the explicit radial function $h_1(x) = -\frac{e^{i\cdot x}}{x}\cdot\left(1+\frac{i}{x}\right)$, $x = k\cdot r$, and simplify, expressing the electric field in terms of the far away electric field strength $E_0 = -E_1$

$$\mathbf{B} = \frac{E_0}{c}\cdot\frac{e^{i\cdot x}}{x}\cdot\left(1+\frac{i}{x}\right)\cdot\sin(\theta)\cdot\mathbf{e}_{\phi} \qquad \mathbf{E} = E_0\cdot\frac{e^{i\cdot x}}{x}\cdot\left[\mathbf{e_r}\cdot 2\cdot\cos(\theta)\cdot\left(\frac{i}{x}-\frac{1}{x^2}\right) + \mathbf{e}_{\theta}\cdot\sin(\theta)\cdot\left(1+\frac{i}{x}-\frac{1}{x^2}\right)\right]$$

There are three interesting regions. The "radiation zone" where waves are pure outgoing that locally have $\mathbf{k}\times\mathbf{E} = \omega\cdot\mathbf{B}$ as in a plane wave, with a direction dependent amplitude, and polarized along meridians.

radiation zone x>>1 $\qquad \mathbf{B} = \frac{E_0}{c}\cdot\frac{e^{i\cdot x}}{x}\cdot\sin(\theta)\cdot\mathbf{e}_{\phi} \qquad \mathbf{E} = E_0\cdot\frac{e^{i\cdot x}}{x}\cdot\sin(\theta)\cdot\mathbf{e}_{\theta}$

The "static zone" contains the static fields characterizing the source, but with the added oscillating time dependence. This includes

static zone x<<1 $\qquad \frac{e^{i\cdot x}}{x}\cdot\left(\frac{i}{x}-\frac{1}{x^2}\right) = -\frac{1}{x^3} + O\left(\frac{1}{x}\right)$

$$\mathbf{B} = 0 \qquad \mathbf{E} = -E_0\cdot\left(\frac{\mathbf{e_r}\cdot 2\cdot\cos(\theta) + \mathbf{e}_{\theta}\cdot\sin(\theta)}{x^3}\right) = \frac{1}{4\cdot\pi\cdot\varepsilon_0}\cdot\frac{3\cdot\mathbf{e_r}\cdot(\mathbf{p}\cdot\mathbf{e_r}) - \mathbf{p}}{r^3} \qquad \mathbf{p} = -\frac{4\cdot\pi\cdot\varepsilon_0\cdot E_0}{k^3}\cdot\mathbf{e_z}$$

The only term left over belongs to the "induction zone" where the field is induced through the time rate of the static component, but not the fully developed radiation.

induction zone x~1 $\quad \mathbf{B} = \frac{E_0}{c}\cdot\frac{i\cdot e^{i\cdot x}}{x^2}\cdot\sin(\theta)\cdot\mathbf{e_\phi} \qquad \mathbf{E} = 0$

Thus, this field corresponds to electric dipole radiation. Incidentally, we have found the solutions to Maxwell's equations that match the fields of an oscillating dipole at the origin!

radiated power

$$\mathbf{S} = \frac{1}{2\cdot\mu_0}\cdot\mathrm{Re}\left(\mathbf{E}\times\overline{\mathbf{B}}\right) = \frac{E_0^{\,2}}{2\cdot\mu_0\cdot c}\cdot\frac{1}{x^2}\cdot\mathrm{Re}\left[\mathbf{e_r}\cdot\sin(\theta)^2\cdot\left(1+\frac{i}{x^3}\right) - \mathbf{e_\theta}\cdot\left[2i\cdot\cos(\theta)\cdot\sin(\theta)\cdot\left(\frac{1}{x}+\frac{1}{x^3}\right)\right]\right]$$

$$\mathbf{S} = \frac{E_0^{\,2}}{2\cdot\mu_0\cdot c}\cdot\frac{\sin(\theta)^2}{x^2}\cdot\mathbf{e_r} = \frac{W_0}{4\cdot\pi\cdot r^2}\cdot\frac{3}{2}\cdot\sin(\theta)^2\cdot\mathbf{e_r} \qquad W_0 = \frac{p^2\cdot\omega^4}{12\cdot\pi\cdot\varepsilon_0\cdot c^3}$$ (total radiated power)

Here the fields are worked out for **dipole TE, m=0**.

Find the vector spherical harmonic...

$$\mathbf{L}\cdot Y_{10} = \frac{1}{\sqrt{2}}\cdot\begin{pmatrix} Y_{1,1}+Y_{1,-1} \\ \dfrac{Y_{1,1}-Y_{1,-1}}{i} \\ 0\end{pmatrix} = i\cdot\sqrt{\frac{3}{4\cdot\pi}}\cdot\sin(\theta)\cdot\begin{pmatrix}-\sin(\phi)\\ \cos(\phi) \\ 0\end{pmatrix} = i\cdot\sqrt{\frac{3}{4\cdot\pi}}\cdot\sin(\theta)\cdot\mathbf{e_\phi}$$

...use for E...

$$\mathbf{E} = i\cdot\sqrt{\frac{3}{4\cdot\pi}}\cdot E_{10}\cdot h_l(k\cdot r)\cdot\sin(\theta)\cdot\mathbf{e_\phi} = E_1\cdot h_l(k\cdot r)\cdot\sin(\theta)\cdot\mathbf{e_\phi}$$

...take the curl for the magnetic field:

$$\mathbf{B} = -\frac{i}{c\cdot k}\cdot\nabla\times\mathbf{E} = -\frac{i\cdot E_1}{c\cdot k}\cdot\left[\mathbf{e_r}\cdot\left[\frac{1}{r\cdot\sin(\theta)}\cdot\frac{\delta}{\delta\theta}\left(\sin(\theta)\cdot h_l(k\cdot r)\cdot\sin(\theta)\right)\right] + \mathbf{e_\theta}\cdot\left[-\frac{1}{r}\cdot\frac{\delta}{\delta r}\left(r\cdot h_l(k\cdot r)\cdot\sin(\theta)\right)\right]\right]$$

$$\mathbf{B} = -\frac{i\cdot E_1}{c}\cdot\left[\mathbf{e_r}\cdot 2\cdot\cos(\theta)\cdot\frac{h_l(k\cdot r)}{k\cdot r} - \mathbf{e_\theta}\cdot\sin(\theta)\cdot\left(h'_1(k\cdot r)+\frac{h_1(k\cdot r)}{k\cdot r}\right)\right]$$

Now put in the explicit radial function $h_1(x) = -\frac{e^{i\cdot x}}{x}\cdot\left(1+\frac{i}{x}\right)$, $x = k\cdot r$, and simplify, expressing the electric field in terms of the far away electric field strength $E_0 = -E_1$

$$\mathbf{E} = E_0 \cdot \frac{e^{i \cdot x}}{x} \cdot \left(1 + \frac{i}{x}\right) \cdot \sin(\theta) \cdot \mathbf{e_\phi} \qquad E_2 = -E_1 \qquad x = k \cdot r$$

$$\mathbf{B} = -\frac{E_0}{c} \cdot \frac{e^{i \cdot x}}{x} \cdot \left[\mathbf{e_r} \cdot 2 \cdot \cos(\theta) \cdot \left(\frac{i}{x} - \frac{1}{x^2}\right) + \mathbf{e_\theta} \cdot \sin(\theta) \cdot \left(1 + \frac{i}{x} - \frac{1}{x^2}\right)\right]$$

radiation zone x>>1 $\quad \mathbf{E} = E_0 \cdot \frac{e^{i \cdot x}}{x} \cdot \sin(\theta) \cdot \mathbf{e_\phi} \qquad \mathbf{B} = -\frac{E_0}{c} \cdot \frac{e^{i \cdot x}}{x} \cdot \sin(\theta) \cdot \mathbf{e_\theta}$

static zone x<<1 $\quad \mathbf{E} = 0 \qquad \mathbf{B} = \frac{E_0}{c} \cdot \left(\frac{\mathbf{e_r} \cdot 2 \cdot \cos(\theta) + \mathbf{e_\theta} \cdot \sin(\theta)}{x^3}\right) = \frac{\mu_0}{4 \cdot \pi} \cdot \frac{3 \cdot \mathbf{e_r} \cdot (\mathbf{m} \cdot \mathbf{e_r}) - \mathbf{m}}{r^3}$

induction zone x~1 $\quad \mathbf{E} = E_0 \cdot \frac{i \cdot e^{i \cdot x}}{x^2} \cdot \sin(\theta) \cdot \mathbf{e_\phi} \qquad \mathbf{B} = 0$

Thus, this field corresponds to magnetic dipole radiation. Incidently, we have found the solutions to Maxwell's equations that match the fields of an oscillating dipole at the origin.

$$\mathbf{m} = \frac{4 \cdot \pi \cdot E_0}{\mu_0 \cdot c \cdot k^3} \cdot \mathbf{e_z}$$

radiated power

$$\mathbf{S} = \frac{1}{2 \cdot \mu_0} \cdot \mathrm{Re}(\mathbf{E} \times \overline{\mathbf{B}}) = \frac{E_0^{\,2}}{2 \cdot \mu_0 \cdot c} \cdot \frac{1}{x^2} \cdot \mathrm{Re}\left[\mathbf{e_r} \cdot \sin(\theta)^2 \cdot \left(1 + \frac{i}{x^3}\right) - \mathbf{e_\theta} \cdot \left[2i \cdot \cos(\theta) \cdot \sin(\theta) \cdot \left(\frac{1}{x} + \frac{1}{x^3}\right)\right]\right]$$

$$\mathbf{S} = \frac{E_0^{\,2}}{2 \cdot \mu_0 \cdot c} \cdot \frac{\sin(\theta)^2}{x^2} \cdot \mathbf{e_r} = \frac{W_0}{4 \cdot \pi \cdot r^2} \cdot \frac{3}{2} \cdot \sin(\theta)^2 \cdot \mathbf{e_r} \qquad W_0 = \frac{\mu_0 \cdot m^2 \cdot \omega^4}{12 \cdot \pi \cdot c^3}$$ (total radiated power)